

IMPORTANCE MEASURES FOR MULTICOMPONENT BINARY SYSTEMS

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Abstract

In this paper we review the theory of importance measures for multicomponent binary systems starting out with the classical Birnbaum measure. We then move on to various time independent measures for systems which do not allow repairs including the Barlow and Proschan measure and the Natvig type 1 measure. For the case with repairs we discuss a measure suggested by Barlow and Proschan along with some new suggestions. We also present some new results regarding importance measures for sets of components. In particular we present a generalization and a new representation of the Natvig type 1 set importance measure. We also indicate how the set measures can be extended to the case with repairs.

1 Introduction

In reliability theory component importance measures are used both in diagnostics and design. By measuring the relative importance of the components, the analyst can determine e.g., which components merit the most additional research and development to improve the overall system reliability. Many different measures of importance have been suggested in the literature, ranging from simple time-dependent measures to weighted measures integrated over time. For a survey of different importance measures, see e.g., Natvig[19].

In the first part of this paper we review a few of these measures. We especially emphasize measures for systems allowing component repairs, where we also suggest some variations or alternative measures. In the second part of the paper we describe briefly how these measures can be estimated using Monte Carlo methods.

The structure of a multicomponent system will be described within the framework of binary monotone systems. The set of components in such a system is denoted by $C = \{1, \dots, n\}$. Let $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))$ denote the component state vector at time t , where $X_i(t) = 1$ if the i th component is functioning at time t and zero otherwise, $i = 1, \dots, n$. The structure function of the system is denoted by ϕ , and is assumed to be a nondecreasing binary function representing the system state as a function of the component state vector, i.e., $\phi = \phi(\mathbf{X}(t))$.

For simplicity we will throughout this paper assume that the components are stochastically independent. The vector of component reliabilities at time t is denoted by $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$, where $p_i(t) = \Pr(X_i(t) = 1)$, $i = 1, \dots, n$.

We also introduce the vector of component unreliabilities at time t , denoted by $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))$, where $q_i(t) = \Pr(X_i(t) = 0)$, $i = 1, \dots, n$.

The system reliability at time t can be expressed in terms of $\mathbf{p}(t)$ as:

$$h = E(\phi(\mathbf{X}(t))) = \Pr(\phi(\mathbf{X}(t)) = 1) = h(\mathbf{p}(t)). \quad (1.1)$$

In the following we also need the following familiar vector notation. Let $\mathbf{z} = (z_1, \dots, z_n)$ be an arbitrary vector of real numbers, and let v be some fixed value. Then:

$$(v_i, \mathbf{z}) = (z_1, \dots, z_{i-1}, v, z_{i+1}, \dots, z_n) \quad (1.2)$$

The above notation is also used to fix values of vector entries corresponding to subsets of the index set. Thus, if S is a subset of C , then (v_S, \mathbf{z}) denotes the vector obtained from \mathbf{z} by fixing the values of all the entries in S by v . More generally, if S_1, \dots, S_s are disjoint subsets of C , and $v^{(1)}, \dots, v^{(s)}$ are real numbers, then $(v_{S_1}^{(1)}, \dots, v_{S_s}^{(s)}, \mathbf{z})$ denotes the vector obtained from \mathbf{z} by fixing the values of all the entries in S_j by $v^{(j)}$, $j = 1, \dots, s$.

A component $i \in C$ is said to be *critical* (for the system) at time t if the following holds:

$$\phi(1_i, \mathbf{X}(t)) - \phi(0_i, \mathbf{X}(t)) = 1. \quad (1.3)$$

Since ϕ is a binary nondecreasing function, it follows that the probability that component i is critical at time t is given by:

$$\Pr(\phi(1_i, \mathbf{X}(t)) - \phi(0_i, \mathbf{X}(t)) = 1) = h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t)). \quad (1.4)$$

Obviously the concept of criticality can be extended to sets of components in many different ways. In this context, however, we say that the set S is *critical* at time t if:

$$\phi(1_S, \mathbf{X}(t)) - \phi(0_S, \mathbf{X}(t)) = 1. \quad (1.5)$$

From this it follows that the probability that set S is critical at time t is given by:

$$\Pr(\phi(1_S, \mathbf{X}(t)) - \phi(0_S, \mathbf{X}(t)) = 1) = h(1_S, \mathbf{p}(t)) - h(0_S, \mathbf{p}(t)). \quad (1.6)$$

We observe that if S is a *cut set*, i.e., $\phi(0_S, \mathbf{X}) = 0$ for all (\cdot_S, \mathbf{X}) , then the probability that S is critical at time t is equal to $h(1_S, \mathbf{p}(t))$. Similarly, if S is a *path set*, i.e., $\phi(1_S, \mathbf{X}) = 0$ for all (\cdot_S, \mathbf{X}) , then the probability that S is critical at time t is equal to $1 - h(0_S, \mathbf{p}(t))$.

2 Importance measures for the no repair case

The following importance measure, known as the Birnbaum measure of the importance of the i th component at time t , was introduced by Birnbaum[5]:

$$I_B^{(i)}(t) = \frac{\partial h(\mathbf{p}(t))}{\partial p_i} = h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t)), \quad i = 1, \dots, n, \quad (2.1)$$

We observe that by (1.4) it follows that $I_B^{(i)}(t)$ is equal to the probability that component i is critical at time t . This observation is often used to obtain a generalization of the Birnbaum measure to situations with dependent components.

One important weakness with this kind of importance measure is that it is restricted to a specific point of time. In order to represent the component importance over a range of time, it is often of interest to develop importance measures which are some sort of weighted average over this range. In general, for given weight functions $w_1(t), \dots, w_n(t)$, such a measure can be expressed as follows:

$$I_w^{(i)} = \int_0^\infty [h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t))] w_i(t) dt, \quad i = 1, \dots, n. \quad (2.2)$$

The weight functions reflect in some suitable sense the *importance* of each point of time. It is also possible to include some sort of discounting effect into the weight function reflecting that the present value of the importance measure may be more important (economically) than the future values. Note that cases where only a bounded interval of time is of interest, say (t_1, t_2) , can also be represented in this fashion by letting the weight functions be zero for all $t \notin (t_1, t_2)$.

A well-known measure of this type is the one suggested in Barlow and Proschan[4]. This measure is defined for systems which do not allow repairs. Denoting the density of the life distribution of the i th component by f_i , $i = 1, \dots, n$, this measure is defined as follows:

$$I_{B-P}^{(i)} = \int_0^\infty [h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t))] f_i(t) dt, \quad i = 1, \dots, n. \quad (2.3)$$

This measure can be interpreted as the probability that the i th component causes system failure when the system eventually fails. For more details see Barlow and Proschan[4].

More sophisticated approaches to this can be found in Natvig[14], Natvig[16], Natvig[17] and Natvig[20]. All the measures suggested in these papers can be written in the form (2.2). As an example we consider the unstandardized version of the Natvig type 1 measure. This measure was motivated by studying the impact on the remaining lifetime of the system when a component fails. However, as shown in Natvig[17] this measure can also be defined as follows: Let T denote the lifetime of a new system, and T_i the lifetime of a new system where the life distribution of the i th component is replaced by the corresponding one where exactly one minimal repair of the component is allowed. That is, $p_i(t)$ is replaced by:

$$p_i(t) + \int_0^t \frac{p_i(u)}{p_i(t-u)} f_i(t-u) du = p_i(t)(1 - \ln p_i(t)). \quad (2.4)$$

Then the importance of the i th component is defined as the increase in expected system lifetime resulting from this, i.e., $ET_i - ET$. Following Natvig[17] it is easy to see that this can be written as:

$$ET_i - ET = \int_0^\infty h([p_i(t)(1 - \ln p_i(t))]_i, \mathbf{p}(t)) dt - \int_0^\infty h(\mathbf{p}(t)) dt, \quad (2.5)$$

assuming the integrals exists. By performing a pivotal decomposition on the i th component we arrive at the following expression for the importance measure:

$$I_{N_1}^{(i)} = \int_0^\infty [h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t))] p_i(t)(-\ln p_i(t)) dt, \quad i = 1, \dots, n. \quad (2.6)$$

Thus, we see that this measure is indeed of the form (2.2) as claimed.

3 Importance measures for systems with repairs

For systems which do allow repairs it is perhaps more natural to consider the limiting or stationary distributions. Thus, instead of component reliabilities, one should consider component availabilities. Under suitable regularity conditions these availabilities are given as the limits of the respective component reliabilities. Thus, we introduce the following quantities:

$$p_i = \lim_{t \rightarrow \infty} p_i(t), \quad i = 1, \dots, n. \quad (3.1)$$

A natural extension of the Birnbaum measure would be the following importance measure:

$$L_B^{(i)} = \lim_{t \rightarrow \infty} I_B^{(i)}(t) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}), \quad i = 1, \dots, n, \quad (3.2)$$

where $\mathbf{p} = (p_1, \dots, p_n)$ is the vector of component availabilities.

Alternatively, Barlow and Proschan[4] defines the importance of the i th component in a system allowing repairs as the stationary probability that the failure of component i is the cause of system failure, given that system failure has occurred. By using renewal theory it can be shown that this measure can be expressed as:

$$L_{B-P}^{(i)} = \frac{[h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})]/(\mu_i + \nu_i)}{\sum_{j=1}^n [h(1_j, \mathbf{p}) - h(0_j, \mathbf{p})]/(\mu_j + \nu_j)}, \quad i = 1, \dots, n, \quad (3.3)$$

where μ_i is the mean life, ν_i the mean repair time of component i , and

$$p_i = \lim_{t \rightarrow \infty} p_i(t) = \frac{\mu_i}{\mu_i + \nu_i}, \quad (3.4)$$

the stationary availability of component i , $i = 1, \dots, n$.

Now, it could be argued that for systems allowing repairs it is not just the points of time when system failures occur that is crucial. It is also of interest to consider which components that contribute the most to the system uptime or downtime periods. This idea leads to two alternative importance measures:

$$L_{UP}^{(i)} = [h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})]p_i, \quad i = 1, \dots, n, \quad (3.5)$$

and

$$L_{DOWN}^{(i)} = [h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})](1 - p_i), \quad i = 1, \dots, n. \quad (3.6)$$

We observe that $L_{UP}^{(i)}$ can be interpreted as the limiting probability that component i is functioning and at the same time critical to the system. In such a state component i is indeed contributing to the system uptime. Similarly, $L_{DOWN}^{(i)}$ can be interpreted as the limiting probability that component i is failed and at the same time critical to the system. In such a state component i is indeed contributing to the system downtime.

By performing a pivotal decomposition on the i th component, $L_{UP}^{(i)}$ and $L_{DOWN}^{(i)}$ can be rewritten as:

$$L_{UP}^{(i)} = h(\mathbf{p}) - h(0_i, \mathbf{p}), \quad i = 1, \dots, n, \quad (3.7)$$

and

$$L_{DOWN}^{(i)} = h(1_i, \mathbf{p}) - h(\mathbf{p}), \quad i = 1, \dots, n. \quad (3.8)$$

Thus, $L_{UP}^{(i)}$ can be interpreted as the reduction in system availability if component i is removed from the system. Similarly, $L_{DOWN}^{(i)}$ can be interpreted as the increase in system availability if component i is replaced by a perfect component. In Aven and Jensen[2] a measure similar to $L_{DOWN}^{(i)}$ is called the *improvement potential* of component i . See also Natvig[17].

Note that by adding (3.7) and (3.8) we see that we have the following relation:

$$L_B^{(i)} = L_{UP}^{(i)} + L_{DOWN}^{(i)} \quad (3.9)$$

Alternatively $L_{UP}^{(i)}$ and $L_{DOWN}^{(i)}$ may be standardized so that they add up to one and expressed in terms of the mean life and repair times as:

$$\tilde{L}_{UP}^{(i)} = \frac{[h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})]\mu_i/(\mu_i + \nu_i)}{\sum_{j=1}^n [h(1_j, \mathbf{p}) - h(0_j, \mathbf{p})]\mu_j/(\mu_j + \nu_j)}, \quad i = 1, \dots, n, \quad (3.10)$$

and

$$\tilde{L}_{DOWN}^{(i)} = \frac{[h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})]\nu_i/(\mu_i + \nu_i)}{\sum_{j=1}^n [h(1_j, \mathbf{p}) - h(0_j, \mathbf{p})]\nu_j/(\mu_j + \nu_j)}, \quad i = 1, \dots, n. \quad (3.11)$$

Example. For a series system with component repair we get that:

$$L_B^{(i)} = \prod_{k \neq i} p_k = \prod_{k \neq i} \frac{\mu_k}{\mu_k + \nu_k} \quad (3.12)$$

$$L_{B-P}^{(i)} = \frac{\prod_{k \neq i} \mu_k}{\sum_{j=1}^n \prod_{k \neq j} \mu_k} \quad (3.13)$$

$$L_{UP}^{(i)} = \prod_{k=1}^n p_k = \prod_{k=1}^n \frac{\mu_k}{\mu_k + \nu_k} \quad (3.14)$$

$$L_{DOWN}^{(i)} = (1 - p_i) \prod_{k \neq i} p_k = \frac{\nu_i \prod_{k \neq i} \mu_k}{\prod_{k=1}^n (\mu_k + \nu_k)} \quad (3.15)$$

$$\tilde{L}_{UP}^{(i)} = 1/n \quad (3.16)$$

$$\tilde{L}_{DOWN}^{(i)} = \frac{\nu_i \prod_{k \neq i} \mu_k}{\sum_{j=1}^n \nu_j \prod_{k \neq j} \mu_k} \quad (3.17)$$

Note that $L_{B-P}^{(i)}$ does not depend on component mean repair times. Furthermore, if we use $L_{UP}^{(i)}$ or $\tilde{L}_{UP}^{(i)}$ all components become equally important.

For a parallel system with component repair we get that:

$$L_B^{(i)} = \prod_{k \neq i} (1 - p_k) = \prod_{k \neq i} \frac{\nu_k}{\mu_k + \nu_k} \quad (3.18)$$

$$L_{B-P}^{(i)} = \frac{\prod_{k \neq i} \nu_k}{\sum_{j=1}^n \prod_{k \neq j} \nu_k} \quad (3.19)$$

$$L_{UP}^{(i)} = p_i \prod_{k \neq i} (1 - p_k) = \frac{\mu_i \prod_{k \neq i} \nu_k}{\prod_{k=1}^n (\mu_k + \nu_k)} \quad (3.20)$$

$$L_{DOWN}^{(i)} = \prod_{k=1}^n (1 - p_k) = \prod_{k=1}^n \frac{\nu_k}{\mu_k + \nu_k} \quad (3.21)$$

$$\tilde{L}_{UP}^{(i)} = \frac{\mu_i \prod_{k \neq i} \nu_k}{\sum_{j=1}^n \mu_j \prod_{k \neq j} \nu_k} \quad (3.22)$$

$$\tilde{L}_{DOWN}^{(i)} = 1/n \quad (3.23)$$

Note that $L_{B-P}^{(i)}$ does not depend on component mean life times. Furthermore, if we use $L_{DOWN}^{(i)}$ or $\tilde{L}_{DOWN}^{(i)}$ all components become equally important.

4 Importance measures for sets

In many practical situations it is of interest to evaluate the importance of a set of components instead of just individual components. It turns out to be many different approaches to this problem. The Birnbaum measure at time t may e.g., be extended to a subset S of the component set C as follows:

$$I_B^{(S)} = h(1_S, \mathbf{p}(t)) - h(0_S, \mathbf{p}(t)). \quad (4.1)$$

According to (1.6) the Birnbaum set importance is equal to the probability that the set is critical. Thus, this definition is a natural generalization of the component importance measure.

Obviously the Birnbaum set importance measure can be used as a starting point for time independent set importance measures as well, paralleling the approach for component important measures in the no repair case. That is, one may consider measures of the form:

$$I_w^{(S)} = \int_0^\infty [h(1_S, \mathbf{p}(t)) - h(0_S, \mathbf{p}(t))] w_S(t) dt, \quad S \subseteq C. \quad (4.2)$$

where $w_S(t)$ is some suitable weight function. This is, however, not the common approach to this problem. Barlow and Proschan[4] introduces an importance measure for minimal cut sets which generalizes their component importance measure. An extension of this measure to general sets can be formulated as follows: The importance of a set $S \subseteq C$ is the probability that the failure of this set (i.e., the point of time when all components in S have failed) coincides with the failure of the system, i.e., the set S “causes” system failure. It is easy to see that this measure can be calculated as follows:

$$I_{B-P}^{(S)} = \sum_{i \in S} \int_0^\infty [h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(t)) - h(0_S, \mathbf{p}(t))] \prod_{j \in S \setminus \{i\}} q_j(t) f_i(t) dt. \quad (4.3)$$

If S contains only a single component, say component i , it is easy to see that $I_{B-P}^{(S)} = I_{B-P}^{(i)}$. Thus, the set measure is indeed a generalization of the component importance measure.

If S is a cut set, $h(0_S, \mathbf{p}(t)) = 0$. Hence, for this case the above formula simplifies to the corresponding expression in Barlow and Proschan[4]. Furthermore, we observe that if S contains a minimal cut set K which is a proper subset of S , then $h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(t)) = 0$ for all $i \in S \setminus K$. Thus, for this case the formula can be written as:

$$I_{B-P}^{(S)} = \sum_{i \in K} \int_0^\infty h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(t)) \prod_{j \in S \setminus \{i\}} q_j(t) f_i(t) dt. \quad (4.4)$$

Finally, note that by the monotonicity of the reliability function h , it follows that:

$$h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(t)) \leq h(1_i, 0_{K \setminus \{i\}}, \mathbf{p}(t)). \quad (4.5)$$

Thus, since we obviously also have that:

$$\prod_{j \in S \setminus \{i\}} q_j(t) \leq \prod_{j \in K \setminus \{i\}} q_j(t), \quad (4.6)$$

we conclude that $I_{B-P}^{(S)} \leq I_{B-P}^{(K)}$. Hence, when looking for the most important cut sets, only the minimal ones need to be considered.

An alternative approach to this problem can be found in Natvig[14] and Natvig[17]. As for the set measure suggested in Barlow and Proschan[4], we extend the definition to general sets. We also restrict ourselves to considering the unstandardized version of the measure. We start out by choosing a subset S of the component set C for which we want to calculate the importance. The Natvig type 1 set importance measure is defined in a fashion similar to the corresponding component importance measure. Thus, we let T denote the lifetime of a new system, and T_S denote the lifetime of a new system where the last component to fail within the set S immediately undergoes a minimal repair as defined in (2.4). Then the importance of set S is defined as the increase in expected system lifetime resulting from this, i.e., $ET_S - ET$. Assuming that all the life distributions are absolutely continuous, it follows that the probability of two components failing simultaneously is zero. Thus, the last component to fail within the set S is almost surely uniquely defined. Assume e.g., that $i \in S$ is the last component within S to fail, and let t be the point of time when this happens. Then the increase in expected system lifetime resulting from a minimal repair of i is given by:

$$\int_0^\infty [h(\frac{p_i(t+z)}{p_i(t)} i, 0_{S \setminus \{i\}}, \mathbf{p}(t+z)) - h(0_S, \mathbf{p}(t+z))] dz \quad (4.7)$$

The contribution from i to the set importance is then found by multiplying the above expression by the probability that all the other components in S are failed at time t and integrating the result with respect to the density of the life distribution of i . That is, we get the following contribution:

$$\int_0^\infty \int_0^\infty [h(\frac{p_i(t+z)}{p_i(t)} i, 0_{S \setminus \{i\}}, \mathbf{p}(t+z)) - h(0_S, \mathbf{p}(t+z))] dz \prod_{j \in S \setminus \{i\}} q_j(t) f_i(t) dt \quad (4.8)$$

We then substitute $s = t + z$ and $u = z$, and perform a pivotal decomposition on the i th component. Finally we add up the contributions from all the components in S . As a result we arrive at the following expression:

$$I_{N_1}^{(S)} = \sum_{i \in S} \int_0^\infty [h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(s)) - h(0_S, \mathbf{p}(s))] \psi_{i,S}(s) ds, \quad (4.9)$$

where we have introduced the weight function $\psi_{i,S}(s)$ defined as follows:

$$\psi_{i,S}(s) = p_i(s) \int_0^s \frac{f_i(s-u)}{p_i(s-u)} \prod_{j \in S \setminus \{i\}} q_j(s-u) du \quad (4.10)$$

We observe that $I_{N_1}^{(S)}$ has a form that is similar to that of $I_{B-P}^{(S)}$, but with a slightly more complicated weight function.

If S contains only a single component, say component i , the weight function simplifies to:

$$\psi_{i,S}(s) = p_i(s)(-\ln p_i(s)). \quad (4.11)$$

By inserting this into (4.9) we see that $I_{N_1}^{(S)} = I_{N_1}^{(i)}$. Thus, the set measure is again a generalization of the component importance measure.

If S is a cut set, $h(0_S, \mathbf{p}(t)) = 0$. Hence, for this case the importance measure simplifies to:

$$I_{N_1}^{(S)} = \sum_{i \in S} \int_0^\infty h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(s)) \psi_{i,S}(s) ds, \quad (4.12)$$

Furthermore, as above, if S contains a minimal cut set K which is a proper subset of S , then $h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(t)) = 0$ for all $i \in S \setminus K$. Thus, only the components in K contributes to the sum, i.e.:

$$I_{N_1}^{(S)} = \sum_{i \in K} \int_0^\infty h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}(s)) \psi_{i,S}(s) ds, \quad (4.13)$$

Finally, note that $K \subseteq S$ implies that:

$$\psi_{i,S} \leq \psi_{i,K}. \quad (4.14)$$

Thus, by using the same monotonicity argument as we did for $I_{B-P}^{(S)}$ we conclude that $I_{N_1}^{(S)} \leq I_{N_1}^{(K)}$. Hence, when looking for the most important cut sets, only the minimal ones need to be considered.

We close this section by briefly considering set measures for systems which allow repairs. We start out by extending the measure suggested by Barlow and Proschan[4] to sets. The natural way to accomplish this is to define the importance of a set S , denoted by $L_{B-P}^{(S)}$, as the limiting probability that the failure of this set coincides with the failure of the system, given that system failure has occurred. By using renewal theory it can be shown that this measure can be expressed as:

$$L_{B-P}^{(S)} = \frac{\sum_{i \in S} [h(1_i, 0_{S \setminus \{i\}}, \mathbf{p}) - h(0_S, \mathbf{p})] [\prod_{j \in S \setminus \{i\}} q_j] / (\mu_i + \nu_i)}{\sum_{j=1}^n [h(1_j, \mathbf{p}) - h(0_j, \mathbf{p})] / (\mu_j + \nu_j)}, \quad (4.15)$$

where $q_j = 1 - p_j$, $j = 1, \dots, n$.

Extending the other measures suggested in the previous section is not so straight-forward. Obviously, a set S may contribute to the system uptime only if all the components in S are functioning. Similarly, S may contribute to the system downtime only if all the components in S are failed. In addition to this we may want to include some criticality condition on the set. As mentioned in the introduction, set criticality may be defined in many different ways. In this context, however, we apply the definition given in (1.5). Thus we arrive at the following two measures:

$$L_{UP}^{(S)} = [h(1_S, \mathbf{p}) - h(0_S, \mathbf{p})] \prod_{i \in S} p_i, \quad (4.16)$$

and:

$$L_{DOWN}^{(S)} = [h(1_S, \mathbf{p}) - h(0_S, \mathbf{p})] \prod_{i \in S} q_i. \quad (4.17)$$

The first measure can be interpreted as the limiting probability that all the components in the set S are functioning, and that the set S is critical. The second measure can be interpreted as the limiting probability that all the components in the set S are failed, and that the set S is critical.

If we restrict ourselves to minimal path or cut sets, even simpler measures may be considered. One may argue that such sets are in a certain sense always critical. Hence, we may skip the criticality factor in the above measures. Obviously a path set typically contributes to the uptime of the system, while a cut set contributes to the downtime. Thus, for the family of minimal path sets of a system, denoted by \mathcal{P} , we define the following importance measure:

$$L_{PATH}^{(P)} = \prod_{i \in P} p_i, \text{ for all } P \in \mathcal{P}. \quad (4.18)$$

Similarly, for the family of minimal cut sets of a system, denoted by \mathcal{K} , we define the following importance measure:

$$L_{CUT}^{(K)} = \prod_{i \in K} q_i, \text{ for all } K \in \mathcal{K}. \quad (4.19)$$

For such families one may want to include some sort of standardization. In the following we will consider two different approaches to standardization: *pre-standardization* and *post-standardization*. The pre-standardized versions can be expressed as follows:

$$\bar{L}_{PATH}^{(P)} = \lim_{t \rightarrow \infty} E\left[\frac{\prod_{i \in P} X_i(t)}{\sum_{Q \in \mathcal{P}} \prod_{i \in Q} X_i(t)}\right], \text{ for all } P \in \mathcal{P}, \quad (4.20)$$

and:

$$\bar{L}_{CUT}^{(K)} = \lim_{t \rightarrow \infty} E\left[\frac{\prod_{i \in K} (1 - X_i(t))}{\sum_{M \in \mathcal{K}} \prod_{i \in M} (1 - X_i(t))}\right], \text{ for all } K \in \mathcal{K}, \quad (4.21)$$

where the expressions within the brackets are redefined to be zero whenever the denominators are zero. An argument for this kind of standardization is that

whenever more than one set contribute to respectively the system up- or downtime, the value of the contributions should be split between the contributors so that they add up to one. Unfortunately this approach does not imply that the resulting importance measures add up to one when summed over all sets in the respective family. The reason for this is that the expressions within the brackets have to be redefined when no set contributes to respectively the system up- or downtime.

The post-standardized versions, on the other hand, are given by:

$$\tilde{L}_{PATH}^{(P)} = \frac{\prod_{i \in P} p_i}{\sum_{Q \in \mathcal{P}} \prod_{i \in Q} p_i}, \text{ for all } P \in \mathcal{P}, \quad (4.22)$$

and:

$$\tilde{L}_{CUT}^{(K)} = \frac{\prod_{i \in K} q_i}{\sum_{M \in \mathcal{K}} \prod_{i \in M} q_i}, \text{ for all } K \in \mathcal{K}. \quad (4.23)$$

We observe that these measure by definition add up to one when summed over all sets in the respective family.

One could argue that by using pre-standardization the resulting measures contain more information about the interrelationship between the path and cut sets. In general, however, we would recommend to use post-standardization. This is also the most common approach to standardization. See e.g., Natvig[17]. The post-standardized measures are very easy to calculate analytically, at least when the families \mathcal{P} and \mathcal{K} are known. The pre-standardized measures, in contrast, require very detailed information about the component state distribution. Still it is of course possible to estimate such measures by using Monte Carlo simulation. See Eisinger[6] and Eisinger[7] for more details about how to do this.

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